Problem 1.

Solve the equation for real numbers: $(x-1)^2(3-x)^2 = 1 + 2(2-x)^2$.

Solution 1:

$$(x-1)^2(3-x)^2 = 1+2(2-x)^2$$

let

$$2-x = y$$

$$3-x = y+1$$

$$1-x = y-1$$

$$(y-1)^2(y+1)^2 = 1+2y^2$$

$$(y^2-1)^2 = 1+2y^2$$

$$y^4-2y^2+1 = 1+2y^2$$

$$y^4-4y^2 = 0$$

$$y^2(y^2-4) = 0$$

$$y = 0, \qquad y = 2, \qquad y = -2$$

$$2-x = 0, \qquad x = 2$$

$$2-x = 2, \qquad x = 2$$

$$2-x = -2, \qquad x = 4$$

Solution 2:

$$\left(x^2 - 2x + \right)\left(9 - 6x + x^2\right) = 1 + 2\left(4 - 4x + x^2\right)$$

$$9x^2 - 18x + 9 - 6x^3 + 12x^2 - 6x + x^4 - 2x^3 + x^2 = 1 + 8 - 8x + 2x^2$$

$$20x^2 - 16x - 8x^3 + x^4 = 0$$

$$x^4 - 8x^3 + 20x^2 - 16x = 0$$

$$x\left(x^3 - 8x^2 + 20x - 16\right) = 0$$

$x = 0 \quad x^3 - 8x^2 + 20x - 16 = 0$

let us try one factor of 16

$$2^3 - 8 \times 2^2 + 20 \times 2 - 16 = 0$$

so 2 is one of the root

$$\begin{array}{r|l} & x^2 - 6x + 8 \\ \hline x - 2 & x^3 - 8x^2 + 20x - 16 \\ & \underline{-x^3 - 2x^2} \\ & \quad -6x^2 + 20x - 16 \\ & \quad \underline{-6x^2 + 12x} \\ & \qquad 8x - 16 \\ & \qquad \underline{8x - 16} \\ & \qquad\quad 0 \end{array}$$

$$x^2 - 6x + 8 = 0$$

$$(x - 4)(x - 2) = 0$$

$x = 4 \quad or \quad x = 2$

so root are 0, 2, 4

Problem 2.

Michael has 100 balls in 10 different colors, 10 balls of each color. Find the smallest n that satisfies the property: if Michael place 100 balls in a circle, in any way, then there are always n adjacent balls which have at least 5 distinct colors.

Solution

Consider the circle in which the 10 balls of one colors are all next to each other. To get at least 5 different colors, you have to take at least 3 of these groups. In total, you need at least

$3 \times 10 + 2 = 32$ adjacent ball. Hence, $n \geq 32$

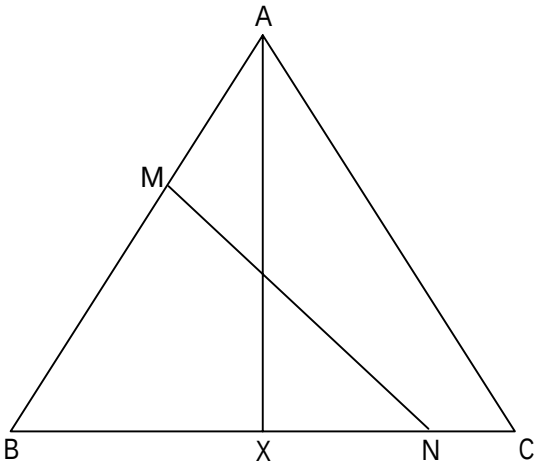
Now we will prove that 32 balls is enough. Consider an arbitrary circle of balls and all possible sets of consecutive balls having exactly 5 colors. (There exists at least one such a set: take an arbitrary ball and add balls on the left one by one, until you have exactly 5 colors). Take such a set having a minimum number of balls. Suppose that the first ball is white. If there is another white ball in the set, then we could have removed the first ball to get a smaller set with the same number of colors, but less balls in total. This contradicts the minimality of our set. Therefore, no other ball is white. In particular, also the last ball of the set is not white; suppose it is black. In the same way, we find that no other ball in the set is black. So there are only one white ball, one black ball, and balls in 3 other colors, at most 10 of each color. In total, there are at most $3 \times 10 + 2 = 32$ balls. Indeed, it is always possible to find a set of 32 consecutive balls in at least 5 different colors. We conclude that the minimum n is 32.

Problem 3.

The triangle ABC has AB = AC and BC is its longest side. The point N is on the side BC and BN = AB. The line perpendicular to AB which passes through N meets AB at M. Prove that the line MN divides both the area and the perimeter of triangle ABC into equal parts.

Solution

Let X be the midpoint of BC.

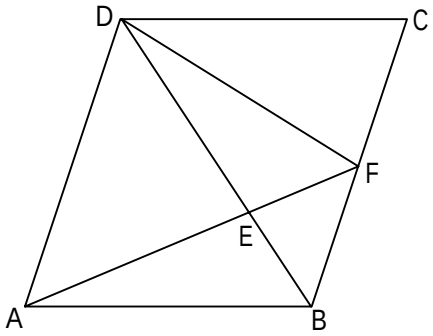


As triangle ABC is isosceles, AX is a line of symmetry of ABC and $\angle AXB = 90^0$. We also know that $\angle NMB = 90^0$ and $\angle ABX$ and that the lengths BN and AB are equal. Therefore triangles AXB and NMB are congruent (by AAS congruence).

As triangle AXB has half the area of ABC, triangle NMB will also. Now for the perimeter: BX = BM so XN = AM. The perimeter of ABC is 2(BX + BA) = 2(BX + BM + AM) = 2(BX + BM + XN) = 2(BN + BM) so MN divides the perimeter into 2.

Problem 4.

A parallelogram ABCD with AD = BD has been given. A point E lies on line segment BD in such a way that AE = DE. The (extended) line AE intersect line segment BC in F. Line DF is the angle bisector of angle CDE. Find the size of ∠ABD



Since AED is an isosceles triangle, angles ∠EDA and ∠DAE are qual.
In turn, these angles are equal to angles ∠EBF and ∠BFE (alternate interior angles). This implies that triangle BFE is isosceles as well, with |EF|.
Comparing triangles ABE and DFE, we see that |AE| = |DE| and |BE| = |FE|. Since ∠BEA and ∠FED are a pair of opposite angles, they have the same size. It follows that triangles ABE and DFE are congruent (SAS).
From the fact that ABE and DFE are congruent it follows that |DF| = |AB|. Since ABCD is a parallelogram, we also have |AB| = |CD|. It follows that triangle CDF is isosceles as well (with vertex D).
On the one hand, this implies that ∠FCD = ∠DFC. On the other hand, we know that triangle DBC is isosceles (since |BD| = |AD| = |BC|, which implies that ∠FCD = ∠CDB = 2∠CDF since DF is the angle bisector of ∠CDB.
Altogether, we have ∠DFC = ∠FCD = 2∠CDF. Since the angles in any triangle sum to 180 degrees, we also know that

180^0 = ∠DFC + ∠FCD + ∠CDF = 5∠CDF.

From this, it follows that ∠CDF = $\frac{1}{5}.180^0 = 36^0$, and hence ∠FCD = 2CDF = 72^0. Since triangle DBC is isosceles, also ∠CDB = 72^0 holds.
Using alternating interior angles, we now find that

∠ABD = ∠CDB = 72^0.

Problem 5.

There are two triples (a, b, c) of positive integers that satisfy the equations

ab + c = 34, and a + bc = 29.

Which two triples are these?

Solution:

$ab + c = 34 \qquad a + bc = 29$

$ab + c - a - bc = 34 - 29$

$ab - a + c - bc = 5$

$a(b - 1) + c(1 - b) = 5$

$a(b - 1) - c(b - 1) = 5$

$(a - c)(b - 1) = 5$

$a - c = 5$ and $b - 1 = 1$

$a = 5 + c \qquad b = 2$

$2(5 + c) + c = 34$

$10 + 2c + c = 34$

$3c = 24$

$c = 8$

a = 13, b = 2, c = 8

$a - c = 1$ and $b - 1 = 5$

$a = c + 1 \qquad b = 6$

$6(c + 1) + c = 34$

$6c + 6 + c = 34$

$7c = 28$

$c = 4$

$a = 5$

a = 5, b = 6, c = 4